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SYNCHRONIZATION THROUGH SYSTEM INTERCONNECTIONS

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1. Introduction

The problem of synchronizing the outputs of N originally not coupled dynamical systems by means of interconnecting feedback paths has a variety of applications:

- In some industrial applications, e.g. steel rolling mills, paper plants, hydraulic press systems (cf. [D'Azzo and Houpis (1966)]) a number of identical machines are employed with identical inputs, and identical outputs are expected, at least asymptotically. The same problem occurs for components of machines (motors, oscillators, generators) and in particular for measuring instruments (output equalization).
- The problem of output equalization is also relevant if the individual systems are not at all identical. For example due to different loading conditions some parameters in principally identical machines may vary (segmented conveyer belts with different loads [Prätzel-Wolters and Schmid (1990)]). Sometimes among a number of nonidentical plants there is one "master plant", and during a transient time the outputs of the other "slave" plants should become identical to the output of the master plant (cf. [Vakilzadeh and Mansour (1990a)]).
- Synchronization of different signals is a problem frequently encountered in electrical engineering and in the field of communication. For example elimination of phase differences ϕ_i between N sinusoidal signals by phase-locked loops (cf. [Unbehauen and Vakilzadeh (1988d)]).
- In neural networks "identical neurons" are interconnected by weight matrices to robustly generate a desired input-output behaviour (tracking problems).

The main reasons why synchronization (output equalization) fails although it is desired are:

- differences in the output initial conditions,
- disturbances in the system signals,
- non-identical, time varying system parameters due to ageing and different operating conditions.

In a series of papers Unbehauen, Vakilzadeh and Mansour considered the problem of output equalization for systems with scalar transfer functions of the form $\frac{k}{s}$, $\frac{k}{s^2}$, $\frac{k_i}{s(s+a)}$ and $\frac{k_i}{s(s+a_i)}$. Their strategy consists in the formation of possible error-signals $e_{ij} = (y_i - y_j)$, the design of controllers $H(s)$ according to the specified input signal class, and loop closing by fully interconnecting the single subsystems (cf. Fig. 1.1).

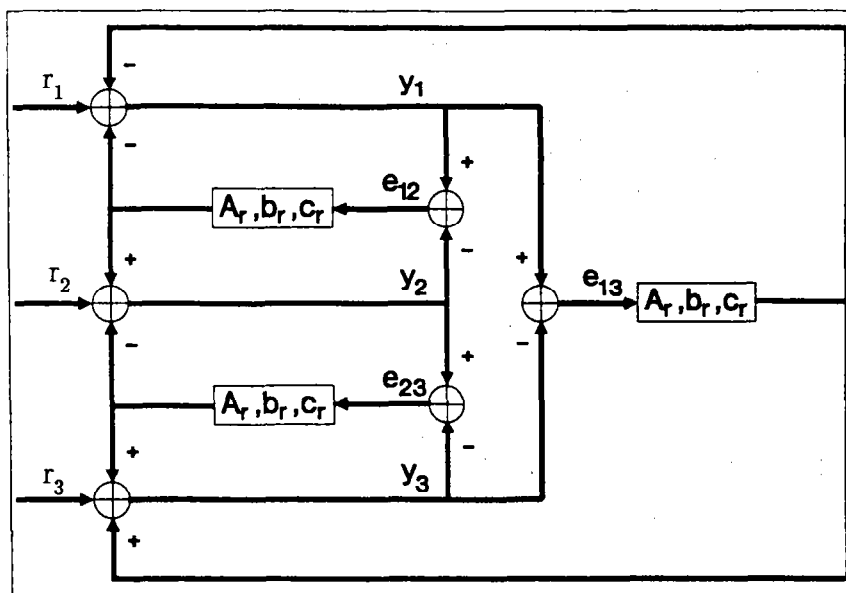


Fig. 1.1: N=3

In Fig. 1.1 the design of $H(s)$ depends on the considered type of polynomial input. (The order of $H(s)$ equals the degree of the rational Laplace transform of the input signal.)

In our paper we pick up the idea of "feed interconnectivity". However, our approach differs in several aspects from the mentioned papers:

- (1) We allow for arbitrary input signals $r_i(\cdot)$ satisfying differential equations of the form: $p_i(D)r_i(\cdot) = 0$, where $p_i(s)$ are arbitrary real polynomials. The associated controllers $H(s)$ in the interconnection loops are generally designed according to the $p_i(s)$, hence it is not necessary to analyse the closed loop system once again when the input class is changed, for example from constant to ramp inputs.

(2) We apply adaptive controllers which can cope not only with different initial conditions and disturbances but also with unknown (time varying) system parameters. However, we restrict the analysis in this paper to minimum phase relative degree one systems with positive high frequency gain. Here it should be mentioned that in [Unbehauen and Vakilzadeh (1988 b),c),d)] simple systems of higher relative degree are considered, involving however, in parts controllers with nonproper transfer functions.

In Section 2 we summarize results which we need in the following analysis of signal synchronization (Section 3) and output equalization (Section 4). Section 2 and 3 are based on [Schmid (1991)] and Section 4 simplifies and improves results contained in this work.

2. Preliminaries on high gain exponential output stabilization

In Section 4 we will construct controllers who eliminate output differences of a number of "similar systems" that are being fed by signals with specified dynamics. The systems belong to the class $\Sigma^+(m)$ of systems (A,B,C) with arbitrary state dimension n and equal number m of inputs and outputs that satisfy the condition:

$$\sigma(CB) \subset \mathbb{C}_+ \quad (2.1)$$

and the minimum phase condition

$$\det \begin{bmatrix} sI-A & -B \\ C & 0 \end{bmatrix} \neq 0 \quad \forall s \in \bar{\mathbb{C}}_+ . \quad (2.2)$$

Those systems are high gain stable in the following sense (cf. [Schmid (1991)], [Ilchmann et al. (1987)], [Mårtensson (1986)]).

Theorem 2.1:

Let $(A,B,C) \in \Sigma^+(m)$. Then the time varying linear system

$$\dot{x}(t) = (A-k(t)BC)x(t) \quad (2.3)$$

is exponentially stable for every

$$k(\cdot) \in S_\infty(\mathbb{R}_+, \mathbb{R}) = \left\{ k(\cdot) \in L_1^{\text{loc}}(\mathbb{R}_+, \mathbb{R}); \lim_{t \rightarrow \infty} k(t) = \infty \right\} .$$

If exponential output stability suffices ($y(t) = Cx(t) \xrightarrow{t \rightarrow \infty} 0$ exponentially) the conditions on (A, B, C) can be relaxed. The system may have unstable zeros, which must be, however, unobservable. To be precise, if $s \in \overline{\mathbb{C}}_+$ is a zero of (A, B, C) , i.e. of $\det \begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix}$, of multiplicity μ , we require

$$\text{rank} \begin{bmatrix} sI - A \\ C \end{bmatrix} \leq n - \mu, \quad (2.4)$$

where n is the state dimension of the system. The class of systems (A, B, C) that have this property and satisfy $\sigma(CB) \subset \mathbb{C}_+$ we denote by $\tilde{\Sigma}^+(m)$. Clearly $\Sigma^+(m) \subset \tilde{\Sigma}^+(m)$.

We note the following

Corollary 2.2:

If $(A, B, C) \in \tilde{\Sigma}^+(m)$ and $k(\cdot) \in S_\infty$, the solutions of

$$\dot{x}(t) = (A - k(t)BC)x(t)$$

satisfy:

$\|Cx(t)\|$ tends to zero exponentially.

Proof:

Let $s_1, \dots, s_k \in \overline{\mathbb{C}}_+$ be the distinct unstable zeros of $(A, B, C) \in \tilde{\Sigma}^+(m)$ with multiplicities μ_1, \dots, μ_k , respectively. There exist μ_i linearly independent solutions $x_1^{(i)}, \dots, x_{\mu_i}^{(i)}$ of

$$\begin{bmatrix} s_i I - A \\ C \end{bmatrix} x = 0, \quad i \in \underline{k}.$$

These solutions are eigenvectors of A and since the s_i are distinct, $\{x_j^{(i)} \mid j \in \underline{\mu_i}, i \in \underline{k}\}$ is linearly independent. We find additional vectors $x_1^{(0)}, \dots, x_{\mu_0}^{(0)}$ such that

$$S := [x_1^{(0)} \dots x_{\mu_0}^{(0)} x_1^{(1)} \dots x_{\mu_1}^{(1)} \dots x_1^{(k)} \dots x_{\mu_k}^{(k)}]$$

is invertible, where $\mu_0 := n - \sum_{i=1}^k \mu_i$.

S obviously transforms A,B,C to

$$S^{-1}AS = \begin{bmatrix} A_1 & 0 \\ A_3 & A_4 \end{bmatrix}, \quad (2.5a)$$

$$CS = [C_1 \quad 0], \quad (2.5b)$$

$$S^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad (2.5c)$$

where A_1 is of size $\mu_0 \times \mu_0$ and the eigenvalues of A_4 are the unstable zeros s_1, \dots, s_k with corresponding multiplicities μ_1, \dots, μ_k , respectively. Since $C_1 B_1 = CB$ and

$$\det \begin{bmatrix} sI-A & -B \\ C & 0 \end{bmatrix} = \det(sI-A_4) \det \begin{bmatrix} sI-A_1 & -B_1 \\ C_1 & 0 \end{bmatrix}$$

$(A,B,C) \in \tilde{\Sigma}^+(m)$ implies $(A_1, B_1, C_1) \in \Sigma^+(m)$. Denoting $x = S \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ we obtain from (2.3)

$$\dot{x}_1 = (A_1 - k(t)B_1C_1)x_1 \quad (2.6)$$

which is exponentially stable by Theorem 2.1. But $C_1 x_1 = Cx$, and the result follows. \square

Let us also note a second corollary needed in Section 4:

Corollary 2.3:

Let $(A,B,C) \in \tilde{\Sigma}^+(m)$ such that the unstable zeros of (A,B,C) are purely imaginary and simple. For any initial values $x(0) = x_0$, $k(0) = k_0$, the system

$$\dot{x}(t) = (A - k(t)BC)x(t) \quad (2.7a)$$

$$\dot{k}(t) = \|Cx(t)\|^2 + \|Cx(t)\| \quad (2.7b)$$

has a unique solution on \mathbb{R}_+ and satisfies

$$x(\cdot) \in L_\infty, \quad (2.8a)$$

$$\lim_{t \rightarrow \infty} \|Cx(t)\| = 0, \quad (2.8b)$$

$\lim_{t \rightarrow \infty} k(t)$ exists and is finite.

(2.8c)

Proof:

A solution exists on some time interval $[0, T], 0 < T \leq \infty$. By Corollary 2.2 the assumption $k(\cdot) \notin L_\infty([0, T])$ implies $Cx(\cdot) \in L_1 \cap L_2$ leads to a contradiction by (2.7b). Thus $k(\cdot) \in L_\infty([0, T])$, and we can assume $T = \infty$. As before, (2.7a) can be decomposed into

$$\dot{x}_1(t) = (A_1 - k(t)B_1C_1)x_1(t)$$

$$\dot{x}_2(t) = (A_3 - k(t)B_2C_1)x_1(t) + A_4x_2(t).$$

By $k(\cdot) \in L_\infty$ we have $C_1x_1(\cdot) = Cx(\cdot) \in L_1$, and since $(A_1, B_1, C_1) \in \Sigma^+(\mu_0)$ this implies $x_1(\cdot) \in L_1$. By the assumption on the unstable zeros of (A, B, C) , for some $M > 0$ we have

$$\|e^{A_4(t-r)}\| \leq M \quad \text{for all } t \geq r \geq 0.$$

Hence, by variations-of-constants,

$$\begin{aligned} \|x_2(t)\| &\leq M\|x_2(0)\| + \int_0^t M\|A_3 - k(r)B_2C_1\| \|x_1(r)\| dr \\ &\leq M\|x_2(0)\| + ML \int_0^t \|x_1(r)\| dr \end{aligned}$$

for some $L > 0$, and $x_1(\cdot) \in L_1$ implies $x_2(\cdot) \in L_\infty$.

□

3. Signal synchronization

Assume that N given signals $r_i(t)$, $i \in \underline{N}^*$, satisfy differential equations:

$$p_i\left(\frac{d}{dt}\right)r_i(t) \equiv 0 \quad (3.1)$$

for some polynomials

*) \underline{N} denotes the set $\{1, \dots, N\}$.

$$p_i(s) = s^{n_i + p_{n_i-1}^{(i)}} s^{n_i-1} + \dots + p_1^{(i)} s + p_0^{(i)} .$$

The problem is to design a common controller (A_r, b_r, c_r) :

$$\begin{aligned} \dot{x}_r &= A_r x_r + b_r u \\ y_r &= c_r x_r \end{aligned} \quad (3.2)$$

such that for the fully interconnected system in Fig. 1.1 the output-signals $y_i(t)$ (modified $r_i(t)$ -signals) get synchronized in the sense that

$$\lim_{t \rightarrow \infty} (y_i(t) - r(t)) = 0 \quad \text{for } i \in \underline{N} \quad (3.3a)$$

where

$$r(t) := \frac{1}{N} \sum_{j=1}^N r_j(t) . \quad (3.3b)$$

Let x_{ij} denote the state of the controller operating on the error signal $e_{ij}(\cdot) := y_j(\cdot) - y_i(\cdot)$. Then the interconnected system is described by

$$\dot{x}_{ij} = A_r x_{ij} + b_r \delta_{ij} (y_j - y_i) \quad (3.4a)$$

$$x_{ij}(0) = x_{ji}(0) , \quad i, j \in \underline{N} , \quad i \neq j \quad (3.4b)$$

$$y_i = r_i + \sum_{\substack{j=1 \\ j \neq i}}^N \delta_{ij} c_r x_{ij} , \quad i \in \underline{N} \quad (3.4c)$$

where

$$\delta_{ij} = \begin{cases} +1 & \text{if } i > j \\ -1 & \text{if } i < j . \end{cases}$$

In the subsequent analysis we only assume

$$\delta_{ij} \in \{+1, -1\} \quad \text{and} \quad \delta_{ij} = -\delta_{ji} \quad \text{for } i, j \in \underline{N}, \quad i \neq j \quad (3.4d)$$

(3.4d) implies for the solution $x_{ij}(\cdot)$ of (3.4a), (3.4b):
 $x_{ij}(\cdot) = x_{ji}(\cdot)$ for all pairs (i,j) , $i \neq j$.

3.1 Theorem:

For every family $(r_1(t), \dots, r_N(t))$ of reference signals satisfying (3.1) the outputs $y_i(t)$ of the interconnection scheme (3.4) satisfy:

$$(i) \quad \lim_{t \rightarrow \infty} (y_i(t) - \frac{1}{N} \sum_{j=1}^N r_j(t)) = 0 \quad \text{for } i \in \underline{N} \quad (3.5)$$

provided: i) $(c_r, A_r) \in \mathbb{R}^{1 \times n} \times \mathbb{R}^{n \times n}$, where $n = \deg \text{lcm}\{p_i(s) \mid i \in \underline{N}\}$ is observable,

$$ii) \quad \det(sI - A_r) = \text{lcm}\{p_i(s) \mid i \in \underline{N}\},$$

$$iii) \quad (A_r - N b_r c_r) \text{ is asymptotically stable.}$$

Proof:

If (c_r, A_r) is observable then $c_r e^{A_r t}$ is a fundamental system of solutions of the differential equation $p(\frac{d}{dt})r(t) = 0$, where $p(s) = \det(sI - A_r)$; hence we can write $r_i(\cdot) = c_r x_{ii}(\cdot)$ for some function $x_{ii}(\cdot)$ satisfying $\dot{x}_{ii} = A_r x_{ii}$. Defining $\delta_{ii} = 1$, $i \in \underline{N}$, (3.4) can be written:

$$\dot{x}_{ij} = A_r x_{ij} + b_r \delta_{ij} (y_j - y_i),$$

$$x_{ij}(0) = x_{ji}(0), \quad i, j \in \underline{N}$$

$$y_i = \sum_{j=1}^N \delta_{ij} c_r x_{ij}$$

and

$$e_{ij} := y_j - y_i = c_r \sum_{\ell=1}^N (\delta_{j\ell} x_{j\ell} - \delta_{i\ell} x_{i\ell}).$$

A simple calculation yields:

$$\dot{e}_{ij} = (A_r - N b_r c_r) e_{ij}, \quad \forall (i, j) \in \underline{N} \times \underline{N}.$$

But $(A_r - Nb_r c_r)$ is assumed to be stable and we obtain:

$$\lim_{t \rightarrow \infty} e_{ij}(t) = \lim_{t \rightarrow \infty} (y_j(t) - y_i(t)) = 0 \quad \forall (i, j) \in \underline{N} \times \underline{N}.$$

Moreover, by (3.4d):

$$\frac{1}{N} \sum_{i=1}^N y_i(t) = \frac{1}{N} \sum_{i=1}^N r_i(t) =: r(t)$$

hence

$$\lim_{t \rightarrow \infty} (y_i(t) - r(t)) = 0 \quad \forall i \in \underline{N}.$$

□

3.2 Remarks:

- (i) The resulting "steady state" signal $r(t)$ equals the average of the input signals $r_i(t)$. In particular, it also satisfies the differential equation $p(\frac{d}{dt})r(\cdot) = 0$.
- (ii) To eliminate the differences between signals with specified dynamics was the concern of the above concept. The resulting steady state signal could be predicted; it incorporated the dynamics of the original signals. If the desired steady state signal $\hat{r}(\cdot)$ is given a priori and the signals $y_i(\cdot)$ are required to approximate $\hat{r}(\cdot)$ asymptotically we have to extend (3.4) into a tracking scheme (cf. [Helmke et al. (1990)]) where the controllers (A_r, b_r, c_r) require the signal $\hat{r}(\cdot)$ as an additional input.

3.3 Construction of (A_r, b_r, c_r) and examples:

The design conditions i)-ii) in Theorem 3.1 can always be satisfied by choice of a suitable (A_r, b_r, c_r) , because no information concerning the concrete signals $r_i(t)$ beside knowledge of the polynomials $p_i(s)$, $i \in \underline{N}$, is required. To be more specific, given:

$$p(s) = \text{lcm}\{p_i(s), i \in \underline{N}\} = s^n + p_{n-1}s^{n-1} + \dots + p_1s + p_0$$

we can always select (c_r, A_r) in observable canonical form:

$$A_r = \begin{bmatrix} 0 & \dots & -p_0 \\ 1 & \ddots & \vdots \\ & \ddots & 0 \\ 0 & \dots & 1 & -p_{n-1} \end{bmatrix}, \quad c_r = [0 \ 0 \ \dots \ 0 \ 1] \quad (3.6a)$$

and

$$b_r = [\hat{p}_0 \ \dots \ \hat{p}_{n-1}]^T \quad (3.6b)$$

such that the polynomial

$$\begin{aligned} \det(sI_n - [A_r - Nb_r c_r]) \\ = s^n + (p_{n-1} + N\hat{p}_{n-1})s^{n-1} + \dots + (p_1 + N\hat{p}_1)s + p_0 + N\hat{p}_0 \end{aligned}$$

is a Hurwitz polynomial.

Assume for example:

$$r_i(t) = A_i \sin(\omega t + \phi_i), \quad i \in \underline{N},$$

i.e. the N signals $r_i(\cdot)$ are sinusoidal signals with same frequency but different amplitudes and phases, then all signals satisfy the same differential equation:

$$((\frac{d}{dt})^2 + \omega^2) r_i(\cdot) = 0$$

hence $p_i(s) = p(s) = s^2 + \omega^2$ for $i \in \underline{N}$.

According to (3.6) we select:

$$A_r = \begin{bmatrix} 0 & -\omega^2 \\ 1 & 0 \end{bmatrix}, \quad c_r = [0 \ 1] \quad \text{and} \quad b_r = [\hat{p}_0 \ \hat{p}_1]^T$$

such that

$$\det[sI_2 - (A_r - Nb_r c_r)] = s^2 + N\hat{p}_1 s + N\hat{p}_0 + \omega^2$$

has stable zeros.

The resulting steady state signal is:

$$r(t) = \frac{1}{N} \sum_{j=1}^N A_i \sin(\omega t + \phi_i)$$

$$= B \sin(\omega t + \theta)$$

where

$$B = \frac{1}{N} \left[\left(\sum_{i=1}^N A_i \cos \phi_i \right)^2 + \left(\sum_{i=1}^N A_i \sin \phi_i \right)^2 \right]^{1/2} \quad (3.7)$$

and

$$\theta = \tan^{-1} \frac{\sum_{i=1}^N A_i \sin \phi_i}{\sum_{i=1}^N A_i \cos \phi_i}.$$

Equation (3.7) can be written in the form:

$$B = \frac{1}{N} \left\{ \sum_{i=1}^N A_i^2 + 2 \sum_{i=1}^{N-1} A_i \sum_{j=i+1}^N A_j \cos(\phi_i - \phi_j) \right\}^{1/2}$$

which shows that the amplitude B of the steady state sinusoidal outputs does not depend on the phase angles ϕ_i themselves, but only on the differences between them. The frequency remains unchanged.

4. System output equalization

We extend now the concept of Section 3 to eliminate differences between outputs of a number of "similar" systems that are being fed by signals with specified dynamics. The resulting steady state output is sought to be somehow related to the open loop outputs of the systems. In this sense the control objective is twofold.

A problem of this type is addressed in [Unbehauen, Vakilzadeh (1989)], generalizing results of [Unbehauen, Vakilzadeh (1988a)]. The authors consider simple-integral systems, i.e. systems with transfer function K/s , and assume identical input signals, constant and ramp functions. In industrial applications it is often very desirable that different samples of an industrial product have identical outputs when the inputs are identical. The context of this paper asks for more general

results. To capture a broad variety of aspects of synchronization we drop the condition of identical inputs and allow in particular for sinusoidal signals. Furthermore, we don't want to assume knowledge of the system parameters. However, fulfilment of the second control objective, maintaining the open loop characteristics, requires that the systems have certain properties in common. Thus one cannot expect results of the generality we obtained in the previous section. As before, we assume the input signals $r_i(\cdot)$, $i \in \underline{N}$ satisfy differential equations

$$p_i \left(\frac{d}{dt} \right) r_i(t) = 0 \quad (4.1)$$

for some monic polynomials $p_i(s)$, $i \in \underline{N}$.

We further assume here that the input signals are bounded. Then the polynomials $p_i(s)$ can be chosen such that they have no zeros in the right-half complex plane and only simple zeros on the imaginary axis.

We construct (A_r, b_r, c_r, d_r) , $d_r=1$, as a minimal state space realization of the transfer function $\frac{q(s)}{p(s)}$, where $p(s) = \text{lcm}\{p_i(s) | i \in \underline{N}\}$ and $q(s)$ is any stable, monic polynomial of the same degree.

This means

(c_r, A_r) is observable,

$$p(s) = \det(sI - A_r) = \text{lcm}\{p_i(s) | i \in \underline{N}\},$$

$$q(s) = \det(sI - A_r + b_r c_r) \neq 0 \quad \text{for all } s \in \overline{\mathbb{C}_+}.$$

The controllers (A_r, b_r, c_r, d_r) , $d_r=1$, are now implemented in the same interconnection architecture as before. The only difference is that the input-signals $r_i(\cdot)$ pass the given system (A_i, b_i, c_i) before they are interconnected through these controllers.

This is shown in the following figure:

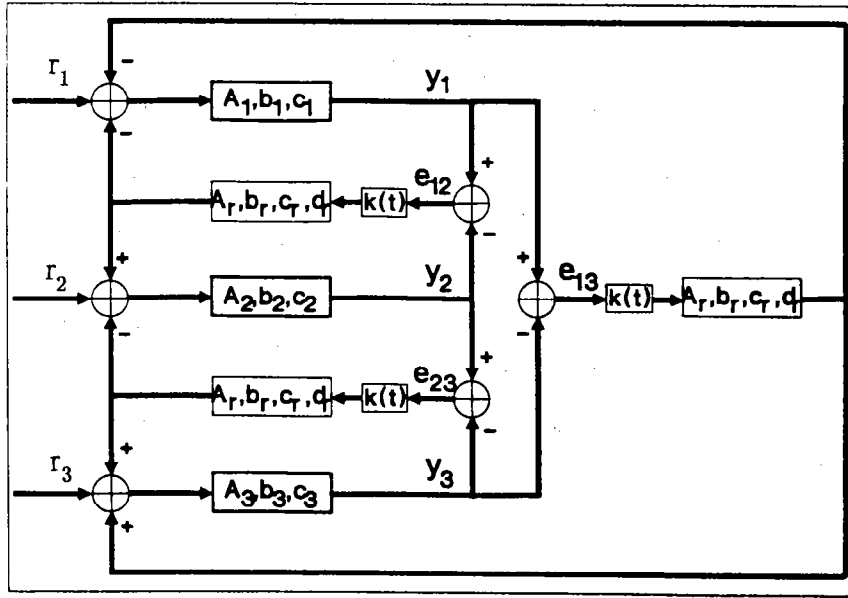


Fig. 4.1

The systems (A_i, b_i, c_i) , $i \in \underline{N}$, are scalar, minimum phase, relative degree one, systems with positive high frequency gain $c_i b_i > 0$, $i \in \underline{N}$. The controllers are single-gain adaptive output feedback controllers.

Theorem 4.1:

Consider N controllable and observable systems $(A_i, b_i, c_i) \in \Sigma^+(1)$ with identical pole polynomials $\alpha(s) = \det[sI - A_i]$, $i \in \underline{N}$. Then for any initial values $x_i^0, x_{ij}^0 = x_{ji}^0, k_i^0, i, j \in \underline{N}$ there exists a unique solution of the interconnected closed loop system

$$\dot{x}_i = A_i x_i + b_i (r_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^N \delta_{ij} (c_r x_{ij} + k(t) \delta_{ij} (y_j - y_i))) \quad (4.2a)$$

$$\dot{x}_{ij} = A_r x_{ij} + b_r k(t) \delta_{ij} (y_j - y_i), \quad i \neq j \quad (4.2b)$$

$$\dot{k} = \sum_{i,j=1}^N ((y_i - y_j)^2 + |y_i - y_j|) \quad (4.2c)$$

$$y_i = c_i x_i \quad (4.2d)$$

on \mathbb{R}_+ . The solution is bounded and satisfies

$$\lim_{t \rightarrow \infty} |y_i(t) - y_j(t)| = 0 \quad \text{for all } i, j \in \underline{N}. \quad (4.3)$$

Moreover, there exist transformations T_{ji} such that the state average $\xi_j(t) := \frac{1}{N} \sum_{i=1}^N T_{ji} x_i(t)$ satisfies

$$\dot{\xi}_j = A_j \xi_j + b_j \left(\frac{1}{N} \sum_{i=1}^N r_i(t) \right). \quad (4.4)$$

Proof:

By the given assumptions we can assume that the systems (A_i, b_i, c_i) are given in observability-canonical form (3.6), in particular $A_i = A_j =: A$ and $c_i = c_j =: c$ for all $i, j \in \underline{N}$. Also, the indices can be rearranged, such that

$$cb_1 \leq cb_i \quad \text{for all } i \in \underline{N}.$$

Furthermore as in Section 3 there exist suitable initial conditions such that

$$r_i(t) = c_r x_{ii}(t) \quad \text{for all } t \in \mathbb{R}_+$$

where

$$\dot{x}_{ii}(t) = A_r x_{ii}(t), \quad x_{ii}(0) = x_{ii}^0.$$

Thus with $\delta_{ii} := 1$ (4.2a,b) reads

$$\dot{x}_i = A x_i + b_i c_r \sum_{j=1}^N \delta_{ij} x_{ij} + k(t) b_i c \sum_{j=1}^N (x_j - x_i) \quad (4.5a)$$

$$(\delta_{ij} \dot{x}_{ij}) = A_r (\delta_{ij} x_{ij}) + k(t) b_r c (x_j - x_i), \quad i, j \in \underline{N}. \quad (4.5b)$$

In order to get a more compact representation of (4.5) we introduce the overall state

$$\underline{x} := (x_1^T \dots x_N^T, \delta_{12} x_{12}^T \dots \delta_{1N} x_{1N}^T, \delta_{23} x_{23}^T \dots \delta_{N-1N} x_{N-1N}^T, x_{11}^T \dots x_{NN}^T)^T.$$

Note that \underline{x} contains only the components $\delta_{ij}x_{ij}$ with $i < j$, because of the identity $\delta_{ij}x_{ij}(t) = -\delta_{ji}x_{ji}(t)$

$$\dim \underline{x}(t) = Nn + \left[\frac{1}{2}N \cdot (N-1) + N \right] n_r .$$

We can write (4.5) in the form:

$$\dot{\underline{x}} = (\underline{A} - k(t)\underline{B}\underline{C})\underline{x}(t) \quad (4.6)$$

where

$$\underline{A} = \begin{bmatrix} \underline{A}_{11} & \underline{A}_{12} & \underline{A}_{13} \\ 0 & \underline{A}_{22} & 0 \\ 0 & 0 & \underline{A}_{33} \end{bmatrix} \quad (4.7)$$

with

$$\underline{A}_{11} = \text{diag}[A, \dots, A]_{Nn \times Nn}$$

$$\underline{A}_{12} =$$

$$\begin{array}{c} \begin{array}{ccc} \overbrace{\hspace{2cm}}^{(N-1)n_r} & \overbrace{\hspace{2cm}}^{(N-2)n_r} & \overbrace{\hspace{1cm}}^{n_r} \\ \left[\begin{array}{ccc|ccc|ccc} b_1 c_r & \dots & b_1 c_r & 0 & \dots & 0 & 0 & \dots & 0 \\ \hline -b_2 c_r & & & b_2 c_r & \dots & b_2 c_r & \vdots & & \vdots \\ & 0 & & \hline & \vdots & & -b_3 c_r & & 0 & \vdots & & 0 \\ & 0 & & & \vdots & & 0 & & \vdots & b_{N-1} c_r \\ & & -b_N c_r & & & -b_N c_r & & & -b_N c_r \end{array} \right] \end{array} \\ =: \left[\begin{array}{ccc|ccc|ccc} \underline{A}_{12}^{(1)} & & & \underline{A}_{12}^{(2)} & & & \dots & & \underline{A}_{12}^{(N-1)} \end{array} \right] \end{array} \quad (4.8)$$

$$\underline{B} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ \vdots & \vdots \\ B_{N+11} & B_{N+12} \end{bmatrix} :=$$

$$\begin{array}{l} N \cdot n \\ (N-1)n_r \\ (N-2)n_r \\ n_r \\ N \cdot n_r \end{array} \left\{ \begin{array}{c} \overbrace{\begin{bmatrix} (N-1)b_1 & -b_1 & \dots & -b_1 \\ -b_2 & (N-1)b_2 & \dots & -b_2 \\ \vdots & \vdots & \ddots & \vdots \\ -b_N & -b_N & \dots & (N-1)b_N \end{bmatrix}}^N \\ \hline \begin{bmatrix} b_r & -b_r & & 0 \\ \vdots & 0 & \ddots & \\ b_r & & & -b_r \end{bmatrix} \\ \hline \begin{bmatrix} 0 & b_r & -b_r & 0 \\ \vdots & \vdots & 0 & \ddots \\ 0 & b_r & & -b_r \end{bmatrix} \\ \hline \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \end{bmatrix} \\ \hline \begin{bmatrix} 0 & & 0 & b_r & -b_r \end{bmatrix} \\ \hline \begin{bmatrix} 0 & & & & 0 \end{bmatrix} \end{array} \right. \quad (4.9)$$

$$\underline{C} = \left[\underbrace{\begin{bmatrix} c & & \\ & \ddots & \\ & & c \end{bmatrix}}_{N \text{ blocks}} \mid \begin{bmatrix} 0 & \dots & 0 \end{bmatrix} \right] =: [\underline{C}_1 \mid 0 \dots 0] \quad (4.10)$$

\underline{A}_{22} , \underline{A}_{33} are block diagonal matrices $\text{diag}(A_r, \dots, A_r)$ with $\frac{1}{2}(N-1)N$ and N blocks, respectively.

$$\underline{A}_{13} = \begin{bmatrix} b_1^{c_r} & 0 & \dots & 0 \\ 0 & b_2^{c_r} & \dots & 0 \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & b_N^{c_r} \end{bmatrix}_{N \cdot n \times N \cdot n_r} \quad (4.11)$$

The transformation

$$S = \text{diag} \left[\overbrace{\begin{bmatrix} I_n & 0 \\ \vdots & \ddots \\ I_n & 0 \dots 0 & I_n \end{bmatrix}}^{N \text{ blocks}}, \overbrace{I_{n_r}, \dots, I_{n_r}}^{(\frac{1}{2}(N-1)N+N) \text{ blocks}} \right] \quad (4.12)$$

transforms \underline{x} to

$$S^{-1} \underline{x} = \begin{bmatrix} x_1^T & (x_2 - x_1)^T, \dots, (x_N - x_1)^T & \delta_{12} x_{12}^T & \dots & \delta_{1N} x_{1N}^T \\ \delta_{23} x_{23}^T & \dots & \delta_{N-1N} x_{N-1N}^T & x_{11}^T & \dots & x_{NN}^T \end{bmatrix}^T \quad (4.13)$$

Partitioning the system matrices correspondingly (\underline{A}_{11} splits into A_{00} and A_{11}), we have

$$S^{-1} A S = \begin{bmatrix} A_{00} & * & * & * \\ 0 & A_{11} & A_{12} & A_{13} \\ 0 & 0 & A_{22} & 0 \\ 0 & 0 & 0 & A_{33} \end{bmatrix} \quad (4.14)$$

where $A_{00} := A$

$A_{11} := \text{diag}(A, \dots, A)_{(N-1)n \times (N-1)n}$

$A_{22} := \underline{A}_{22}, \quad A_{33} := \underline{A}_{33}$

$$A_{12} := \left[\begin{array}{cccc|c} -b_1 c_r - b_2 c_r & -b_1 c_r & \dots & -b_1 c_r & A_{12}^{(2)}, \dots, A_{12}^{(N-1)} \\ -b_1 c_r & -b_1 c_r - b_3 c_r & \dots & -b_1 c_r & \\ \vdots & & & & \\ -b_1 c_r & \dots & \dots & -b_1 c_r - b_N c_r & \end{array} \right] \quad (4.15)$$

$A_{12}^{(2)}, \dots, A_{12}^{(N-1)}$ are obtained from $A_{12}^{(2)}, \dots, A_{12}^{(N-1)}$ by deleting the first row in each.

$$A_{13} := \left[\begin{array}{cccc|c} b_1 c_r & 0 & \dots & 0 & \\ -b_1 c_r & b_2 c_r & 0 & \dots & 0 \\ -b_1 c_r & 0 & \cdot & \cdot & 0 \\ \vdots & & \cdot & \cdot & \\ -b_1 c_r & 0 & \dots & 0 & b_N c_r \end{array} \right] \quad (4.16)$$

In $S^{-1}\underline{B}$ the matrix block $[B_{11} : B_{12}]$ splits into 4 subblocks

$$\left[\begin{array}{c|c} B_{11}^{(1)} & B_{12}^{(1)} \\ \hline B_{11}^{(2)} & B_{12}^{(2)} \end{array} \right]. \text{ The other blocks of } \underline{B} \text{ remain invariant:}$$

$$S^{-1}\underline{B} = \left[\begin{array}{c|c} \begin{array}{cccc} (N-1)b_1 & -b_1 & -b_1 & \dots & -b_1 \\ -b_2 - (N-1)b_1 & (N-1)b_2 + b_1 & -b_2 + b_1 & \dots & -b_2 + b_1 \\ -b_3 - (N-1)b_1 & -b_3 + b_1 & (N-1)b_3 + b_1 & \dots & -b_3 + b_1 \\ \vdots & \vdots & & & \vdots \\ -b_N - (N-1)b_1 & -b_N + b_1 & \dots & \dots & (N-1)b_N + b_1 \end{array} & \begin{array}{c} B_{21} \\ B_{22} \\ \vdots \\ B_{N+1 \ 1} \end{array} \\ \hline \begin{array}{c} B_{21} \\ B_{22} \\ \vdots \\ B_{N+1 \ 1} \end{array} & \begin{array}{c} B_{22} \\ \vdots \\ B_{N+1 \ 2} \end{array} \end{array} \right] \quad (4.17)$$

and

$$\underline{C} S = \left[\begin{array}{c|c} \begin{array}{cccc} c & 0 & \dots & 0 \\ \hline c & c & & \\ \vdots & \cdot & \cdot & \\ c & & & c \end{array} & \begin{array}{c} 0 \\ \dots \\ 0 \end{array} \end{array} \right] =: \left[\begin{array}{c|c} \begin{array}{c} c \\ \hline c \\ \vdots \\ c \end{array} & \begin{array}{c} 0 \\ \vdots \\ C_1 \end{array} \end{array} \right] \begin{array}{c} 0 \\ \dots \\ 0 \end{array}$$

Next we select the new system state

$$x := ((x_2 - x_1)^T \dots (x_N - x_1)^T \mid \delta_{12} x_{12}^T \dots \delta_{N-1N} x_{N-1N}^T \mid x_{11}^T \dots x_{NN}^T)^T \quad (4.18)$$

This is due to the fact that we want to show $\lim_{t \rightarrow \infty} |y_i(t) - y_j(t)|$ as a consequence of asymptotic stability of the state space system $(\tilde{A}, \tilde{B}, \tilde{C})$ associated to the state (4.18).

By (4.6) $x(t)$ solves

$$\dot{x}(t) = (\tilde{A} - k(t)\tilde{B}\tilde{C})x(t), \quad (4.19)$$

where

$$\tilde{A} := \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} B_{12}^{(2)} \\ \hline B_{22} \\ \vdots \\ B_{N+12} \end{bmatrix}, \quad \tilde{C} = [C_1 \quad 0 \quad \dots \quad 0]$$

For system (4.19) we verify the assumptions of Corollary 2.2. To check the condition on the zeros of $(\tilde{A}, \tilde{B}, \tilde{C})$ we first determine

$$\underline{D}(s) := \det \begin{bmatrix} sI - \tilde{A} & -\tilde{B} \\ \tilde{C} & 0 \end{bmatrix}.$$

We have

$$\begin{aligned} D(s) &= \det[sI - \tilde{A}] \det[\tilde{C}(sI - \tilde{A})^{-1} \tilde{B}] \\ &= \det[sI - A_{11}] \det[sI - A_{22}] \det[sI - A_{33}] \\ &\quad \cdot \det[C_1 \cdot [(sI - A_{11})^{-1}, (sI - A_{11})^{-1} A_{12} (sI - A_{22})^{-1}, \\ &\quad (sI - A_{11})^{-1} A_{13} (sI - A_{33})^{-1}] \cdot \tilde{B}] \end{aligned}$$

Now let $\tilde{B} = \begin{bmatrix} B_1 \\ -I \\ B_2 \\ -I \\ B_3 \end{bmatrix}$ be the decomposition of \tilde{B} corresponding to

the decomposition of \tilde{A} , where B_1 is the submatrix of \tilde{B} formed by the first $(N-1) \cdot n$ rows, B_2 the submatrix formed by the next $[(N-1)+(N-2)+ \dots +1] \cdot n_r$ rows and B_3 the matrix formed by the last $N \cdot n_r$ rows of \tilde{B} . An easy calculation shows that

$$C_1 [sI - A_{11}]^{-1} B_1 = \begin{bmatrix} (N-1)\tilde{\beta}_2 & -\tilde{\beta}_2 & \dots & -\tilde{\beta}_2 \\ -\tilde{\beta}_3 & (N-1)\tilde{\beta}_3 & \dots & -\tilde{\beta}_3 \\ \vdots & \vdots & \ddots & \vdots \\ -\tilde{\beta}_N & \dots & \dots & (N-1)\tilde{\beta}_N \end{bmatrix} + \begin{bmatrix} \tilde{\beta}_1 & \dots & \tilde{\beta}_1 \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \tilde{\beta}_1 & \dots & \tilde{\beta}_1 \end{bmatrix}$$

where

$$\tilde{\beta}_i := c[sI - A]^{-1} b_i, \quad i=1, \dots, N$$

and

$$C_1 [sI - A_{11}]^{-1} A_{12} [sI - A_{22}]^{-1} B_2 = \begin{bmatrix} (N-1)\tilde{\beta}_2 g(s) & -\tilde{\beta}_2 g(s) & \dots & -\tilde{\beta}_2 g(s) \\ -\tilde{\beta}_3 g(s) & (N-1)\tilde{\beta}_3 g(s) & \dots & -\tilde{\beta}_3 g(s) \\ \vdots & \vdots & \ddots & \vdots \\ -\tilde{\beta}_N g(s) & -\tilde{\beta}_N g(s) & \dots & (N-1)\tilde{\beta}_N g(s) \end{bmatrix} + \begin{bmatrix} \tilde{\beta}_1 g(s) & \dots & \tilde{\beta}_1 g(s) \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \tilde{\beta}_1 g(s) & \dots & \tilde{\beta}_1 g(s) \end{bmatrix}$$

with

$$g(s) := c_r [sI - A_r]^{-1} b_r.$$

Hence, noting that $B_3=0$, we obtain:

$$\begin{aligned} \underline{D}(s) &= \det(sI-A)^{N-1} \det[sI-A_r]^{\frac{1}{2}N(N+1)} \\ &\quad \cdot \det[sI-A]^{-(N-1)} \det(L(s))(1+g(s))^{N-1} \end{aligned} \quad (4.19)$$

where

$$L(s) := \begin{bmatrix} (N-1)\beta_2+\beta_1 & -\beta_2+\beta_1 & \dots & -\beta_2+\beta_1 \\ -\beta_3+\beta_1 & (N-1)\beta_3+\beta_1 & \dots & -\beta_3+\beta_1 \\ \vdots & \vdots & & \\ -\beta_N+\beta_1 & -\beta_N+\beta_1 & \dots & (N-1)\beta_N+\beta_1 \end{bmatrix}$$

and

$$\beta_i = c \operatorname{adj}[sI-A] b_i, \quad i=1, \dots, N.$$

With the notations in (4.1) we obtain from (4.19):

$$\underline{D}(s) = p(s)^{\frac{1}{2}N(N+1)-(N-1)} q(s)^{N-1} \cdot \det L(s) \quad (4.20)$$

By the Appendix-lemma:

$$\det L(s) = N^{N-2} \sum_{i=1}^N \prod_{\substack{j=1 \\ j \neq i}}^N \beta_j$$

This result is obtained by setting:

$$\alpha_i := -\beta_i + \beta_1, \quad i \in \underline{N}$$

$$\beta_1 := \frac{1}{N};$$

for $i \geq 2$ the β_i 's in the Lemma coincide with the β_i 's in $L(s)$. As all summands $\prod_{j \neq i} \beta_j$ above are by assumption stable polynomials of the same degree with positive leading coefficients, this implies that $\det L(s)$ itself is stable.

To apply Corollary 2.2 we verify that $(\tilde{A}, \tilde{B}, \tilde{C}) \in \tilde{\Sigma}^+(M)$,

$$M := \underline{(N-1)n + \frac{1}{2}N(N+1)n_r} :$$

By (4.20), an unstable zero s of $(\tilde{A}, \tilde{B}, \tilde{C})$ is a (simple) zero of $p(s)$, and is therefore of multiplicity $\frac{1}{2}N(N+1)-(N-1)$.

But

$$\text{rank} \begin{bmatrix} sI - \tilde{A} \\ \tilde{C} \end{bmatrix} = \text{rank} \begin{bmatrix} sI - A_{11} & -A_{12} & -A_{13} \\ 0 & sI - A_{22} & 0 \\ 0 & 0 & sI - A_{33} \\ C_1 & 0 & 0 \end{bmatrix}$$

$$\leq (M+N-1) - \left[\frac{1}{2}(N-1)N + N \right]$$

$$= M - \left[\frac{1}{2}N(N+1) - (N-1) \right].$$

So it remains to show that $\sigma(\tilde{C}\tilde{B}) \subset \mathbb{C}_+$:

If λ is an eigenvalue of

$$\tilde{C}\tilde{B} = \begin{bmatrix} (N-1)cb_2 + cb_1 & -cb_2 + cb_1 & \dots & -cb_2 + cb_1 \\ -cb_3 + cb_1 & (N-1)cb_3 + cb_1 & & -b_3 + b_1 \\ \vdots & & \ddots & \vdots \\ -cb_N + cb_1 & \dots & & (N-1)cb_N + cb_1 \end{bmatrix}$$

then by Gershgorin's Theorem we have for some $j \in \underline{N}$, $j \geq 2$:

$$(N-1)cb_j + cb_1 - \text{Re} \lambda \leq |(N-1)cb_j + cb_1 - \lambda| \leq (N-2)(cb_j - cb_1) \quad (4.21)$$

since $cb_j - cb_1 > 0$ was assumed.

(4.21) implies $\text{Re} \lambda > 0$, and Corollary 2.2 can be applied.

By virtue of this corollary, since

$$Cx = \begin{bmatrix} y_2 - y_1 \\ \vdots \\ y_N - y_1 \end{bmatrix}$$

the assumption $k(\cdot) \notin L_\infty([0, T])$ implies $|y_i(\cdot) - y_j(\cdot)| \in L_1 \cap L_2$ for all $i, j \in \underline{N}$, and (4.2c) gives a contradiction.

Thus, $k(\cdot) \in L_\infty([0, T])$ and the solution of

$$\dot{x}(t) = (A - k(t)BC)x(t)$$

$$k(t) = \sum_{i,j=1}^N ((y_i(t) - y_j(t))^2 + |y_i(t) - y_j(t)|)$$

with arbitrary initial value (x_0, k_0) extends to \mathbb{R}_+ and is unique. By solving (4.2a) for $i=1$ provides us with a unique closed loop solution. Boundedness of the solution and (4.3) follows as in the proof of Corollary 2.3.

Finally, let S_i transform (A_i, b_i, c_i) to controllability canonical form:

$$S_i^{-1} A_i S_i = A_c$$

$$S_i^{-1} b_i = b_c$$

From (4.2a) we have

$$(S_i^{-1} x_i) = A_c (S_i^{-1} x_i) + b_c r_i(t) + b_c \sum_{\substack{j=1 \\ j \neq i}}^N (c_r \delta_{ij} x_{ij} + k(t)(y_j - y_i)) .$$

Since $\delta_{ij} x_{ij} = -\delta_{ji} x_{ji}$ for $i \neq j$ this implies

$$\left(\frac{1}{N} \sum_{i=1}^N S_i^{-1} x_i \right) = A_c \left(\frac{1}{N} \sum_{i=1}^N S_i^{-1} x_i \right) + b_c \sum_{i=1}^N r_i(t) ,$$

hence

$$\left(\frac{1}{N} \sum_{i=1}^N S_j S_i^{-1} x_i \right) = A_j \left(\frac{1}{N} \sum_{i=1}^N S_j S_i^{-1} x_i \right) + b_j \left(\frac{1}{N} \sum_{i=1}^N r_i(t) \right) .$$

□

A p p e n d i x

The following technical lemma is needed in section 4.

Lemma:

Let $\alpha_i, \beta_i \in \mathbb{R}[s]$, $i \in \underline{N}$, $N \in \mathbb{N}$. Then:

$$\begin{aligned}
 D &:= \det \begin{bmatrix} \alpha_1 + N\beta_1 & \alpha_1 & \alpha_1 & \dots & \alpha_1 \\ \alpha_2 & \alpha_2 + N\beta_2 & \alpha_2 & \dots & \alpha_2 \\ \alpha_3 & \alpha_3 & \alpha_3 + N\beta_3 & \dots & \alpha_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_N & \alpha_N & \alpha_N & & \alpha_N + N\beta_N \end{bmatrix} \\
 &= N^N \prod_{j=1}^N \beta_j + N^{N-1} \sum_{i=1}^N \alpha_i \prod_{\substack{j=1 \\ j \neq i}}^N \beta_j
 \end{aligned}$$

Proof:

Subtract column 1 from the columns 2, ..., N of the above matrix and calculate the determinant with respect to the first column:

$$\begin{aligned}
 D &= \det \begin{bmatrix} \alpha_1 + N\beta_1 & -N\beta_1 & -N\beta_1 & \dots & -N\beta_1 \\ \alpha_2 & N\beta_2 & 0 & & 0 \\ \alpha_3 & 0 & N\beta_3 & & \\ \vdots & \vdots & & \ddots & \\ \alpha_N & 0 & & & N\beta_N \end{bmatrix} \\
 &= (\alpha_1 + N\beta_1) N^{N-1} \prod_{j \neq 1} \beta_j + N^{N-1} \sum_{i=2}^N \alpha_i \prod_{j \neq i} \beta_j \\
 &= N^N \prod_{j=1}^N \beta_j + N^{N-1} \sum_{i=1}^N \alpha_i \prod_{j \neq i} \beta_j
 \end{aligned}$$

□

References

D'Azzo and Houpis, C.H. (1966): *Feedback Control System analysis and Synthesis*, McGraw Hill, N.Y.

Helmke, U., Prätzel-Wolters, D. and Schmid, S. (1990): *Adaptive tracking for scalar minimum phase systems; Control of Uncertain Systems* (eds. Hinrichsen, D. and Mårtensson, B.), Series Progress in Systems and Control Theory, Birkhäuser, Boston

Ilchmann, A., Owens, D.H. and Prätzel-Wolters, D. (1987a): *High gain robust adaptive controllers for multivariable systems*; Syst. Control Lett. 8, 397-404

Mårtensson, B. (1986): *Adaptive Stabilization*; Doctoral Dissertation, Lund Institute of Technology, Lund

Prätzel-Wolters, D. and Schmid, S. (1991): *Adaptive speed synchronization of segmented conveyor belts*; in: Proc. ECMI '90 Conf., Lahti

Schmid, S. (1991): *Adaptive Synchronization of Interconnected Systems*, PHD-thesis, University of Kaiserslautern, Germany

Unbehauen, H. and Vakilzadeh, I. (1988a): *Automatic error elimination between outputs of 'n' identical simple-integral plants with different initial conditions*; International Journal of Control, Vol. 47 (4), 1101-1116

Unbehauen, H. and Vakilzadeh, I. (1988c): *Optimal synchronization of 'n' identical double-integral plants with identical inputs but different initial conditions*; Optimal Control Applications & Methods, Vol. 9 (4), 371-401

Unbehauen, H. and Vakilzadeh, I. (1988d): *Automatic amplitude and phase equalization of 'n' sinusoidal signals of the same frequency*; Journal of the Franklin Institute, Vol. 325 (3), 369-383

Unbehauen, H. and Vakilzadeh, I. (1989): *Synchronization of 'n' non-identical simple-integral plants*; International Journal of Control, Vol. 50 (2), 543-574

Vakilzadeh, I. and Mansour, M. (1990a): *Synchronization of 'n' integral-plus-double time constant plants with non-identical gain and time constants*; Journal of the Franklin Institute, Vol. 327 (4), 579-593

Vakilzadeh, I. and Mansour, M. (1990b): *Automatic amplitude, phase and frequency equalization of 'n' sinusoidal signals*; Journal of the Franklin Institute, Vol. 327 (4), 595-600